Second semestral exam 2006
M.Math I — Algebra II
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Be very brief. If you need to use a standard result without proof, state it clearly. The first question carries 24 marks and the others carry 10 marks each. Any score > 100 *will count as* 100.

Q 1.
Are the following statements true or false ? Answer with one single line of justification each. The notations used are as follows. A is a commutative ring with unity $S \subset A$ is a multiplicatively closed subset containing 1, M is an A-module. ζ_n is a primitive n-th root of unity.
(a) Each element of $S^{-1}M$ can be expressed in the form $\frac{1}{s} \otimes m$.
(b) An algebraic number α all of whose conjugates have absolute value 1 must be a root of unity.
(c) The angle of 7 degrees is constructible by a ruler and a compass.
(d) Every polynomial of degree 4 over $\mathbf{Q}$ is solvable by radicals.
(e) The ring $\mathbf{Z}[\sqrt{5}]$ is integrally closed.
(f) Over $\mathbf{Z}[X]$ there exists a free module with a submodule which is not free.

Q 2.
Let A be a PID with quotient field K. If $A \subset B \subset K$, for an intermediate subring B, show that B must be a PID as well.

Q 3.
Let M be a finitely generated module over a commutative ring A with unity. Using the NAK lemma or otherwise, prove that any surjective A-module homomorphism $\theta : M \to M$ is automatically an isomorphism.

OR

Let $A \subset B$ be domains and let C denote the integral closure of A in B. Suppose $f, g \in B[X]$ are monic polynomials such that $fg \in C[X]$, prove that $f, g \in C[X]$.

Q 4.
Let A be a commutative ring with unity. Assume that every prime ideal is principal. Show that every ideal must be principal.

OR

Let A be a domain and K, its quotient field. Let L/K be a finite, Galois extension and B, the integral closure of A in L. If P is a prime ideal of P, prove that $\mathrm{Gal}(L/K)$ acts transtively on the set of prime ideals of B which lie over P.

Q 5.
Let K be any field and $a \in K$. Suppose m, n are relatively prime. Prove that $X^{mn} - a$ is irreducible over K if, and only if, both $X^m - a$ and $X^n - a$ are.

Q 6.
Find the partial fraction decomposition of $\frac{1}{x^n-1}$ over $\mathbf{Q}$.

Q 7.
Find a Galois extension of $\mathbf{Q}$ whose Galois group cyclic of order 13.

Q 8.
If $f \in \mathbf{F}_p[X]$ is a product of irreducible factors of degrees $d_1, d_2, \cdots, d_r$, then show that the splitting field of f over $\mathbf{F}_p$ has degree $\mathrm{LCM}(d_1, d_2, \cdots, d_r)$.

OR

Let $\alpha \in \overline{\mathbf{F}_p}$, the algebraic closure of $\mathbf{F}_p$.
Prove that $[\mathbf{F}_p(\alpha) : \mathbf{F}_p] = \min \{n : \alpha^{\frac{p^n-1}{p-1}} \in \mathbf{F}_p\}$.

Q 9.
Let $f \in K[X]$ be irreducible, K any field. If α is a root of f, and $\deg f = 15$, show that f cannot decompose over $K(\alpha)$ as
$f = (\deg 1)(\deg 1)(\deg 1)(\deg 2)(\deg 2)(\deg 8)$ in $K(\alpha)[X]$.

OR

Find the number of fields intermediate between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt{2}, \sqrt{3})$. Prove your assertion.